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Free groups of the special orthogonal groups

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In 1924, Banach and Tarski proved a surprise theorem which can enlarge subsets of the Euclidean space.

The Hausdorff-Banach-Tarski paradox. [BaT; W: Th.3.11]

$n \geqq 3$: integer, U, $V \subsetneqq \mathbb{R}^n$: bdd, $\mathrm{int}\, U \neq \emptyset$, $\mathrm{int}\, V \neq \emptyset$
$\Rightarrow$ *$^\exists \ell$: positive integer,*
$^\exists U_0$, $^\exists U_1, \ldots, {}^\exists U_{\ell-1} \subsetneqq U$: pairwise disjoint,
$^\exists V_0$, $^\exists V_1, \ldots, {}^\exists V_{\ell-1} \subsetneqq V$: pairwise disjoint,
$^\exists \gamma_0$, $^\exists \gamma_1, \ldots, {}^\exists \gamma_{\ell-1} \in SG_n(\mathbb{R})$ such that

$$U = \bigcup_{i=0}^{\ell-1} U_i, \qquad V = \bigcup_{i=0}^{\ell-1} V_i \qquad \text{and} \qquad \gamma_i(U_i) = V_i \quad \text{for} \quad i = 0, 1, \ldots, \ell - 1,$$

where $SG_n(\mathbb{R})$ is the group of all orientation-preserving isometries of $\mathbb{R}^n$.

Remark. This paradox is proved by using the axiom of choice.

Let X be a non-empty set and G a group acting on X (denoted by $G \curvearrowright X$). It is essential for the proof of such a paradox for X and G, to prove the existence of a free subgroup of rank 2 of G,

$$F_2 = \langle \alpha, \beta \rangle = (\text{the group generated by } \alpha \text{ and } \beta) = \{w : \text{reduced word in } \alpha^{-1}, \beta^{-1}, \alpha, \beta\}.$$

The group F_2 is partitioned into five disjoint subsets:

$$F_2 = \{\mathrm{id}\} \cup W_{\alpha^{-1}} \cup W_{\beta^{-1}} \cup W_\alpha \cup W_\beta,$$

where $W_\lambda = \{w \in F_2 : w \text{ begins on the left with } \lambda\}$. Then, F_2 is constructed by two sets of above in two ways:

$$F_2 = \alpha W_{\alpha^{-1}} \cup W_\alpha \qquad \text{and} \qquad F_2 = \beta W_{\beta^{-1}} \cup W_\beta.$$

The group F_2 enables us to duplicate subsets of a set on which it acts, so it is useful to prove the Hausdorff-Banach-Tarski paradox. For a subgroup $H \subsetneqq G$, the action $H \curvearrowright X$ is said to be

$$\textit{without fixed points} \underset{\text{def}}{\Leftrightarrow} {}^\forall w \in H \setminus \{\mathrm{id}\},\ \neg^\exists x \in X \text{ s.t. } w(x) = x,$$
$$\textit{locally commutative} \underset{\text{def}}{\Leftrightarrow} ({}^\forall w, w' \in H \setminus \{\mathrm{id}\},\ ({}^\exists x \in X \text{ s.t. } w(x) = x = w'(x)) \Rightarrow ww' = w'w).$$

The motivation of considering the existence of a free group whose action is "without fixed points" or "locally commutative" is the following.

Proposition. [Dek1; W: Cor.4.12 & Th.4.5]
Let $F_2 \subsetneqq G$ be a free subgroup of rank 2. Then,

the action $F_2 \curvearrowright X$ is locally commutative
$\Rightarrow$ *$^\exists A_0, {}^\exists A_1, {}^\exists A_2, {}^\exists A_3 \subsetneqq X$: pairwise disjoint,*
$^\exists B_0, {}^\exists B_1 \subsetneqq X$: pairwise disjoint,
$^\exists B_2, {}^\exists B_3 \subsetneqq X$: pairwise disjoint, such that

$$X = A_0 \cup A_1 \cup A_2 \cup A_3 = B_0 \cup B_1 = B_2 \cup B_3 \quad \text{and} \quad A_i \approx_{F_2} B_i \quad \text{for} \quad i = 0, 1, 2, 3,$$

where $K \approx_H L \underset{\text{def}}{\Leftrightarrow} {}^\exists\gamma \in H$ s.t. $\gamma(K) = L$. Moreover,

the action $F_2 \curvearrowright X$ is without fixed points
$\Rightarrow$ *$^\exists A, {}^\exists B, {}^\exists C \subsetneqq X$: pairwise disjoint, such that*

$$X = A \cup B \cup C \qquad \text{and} \qquad A \approx_{F_2} B \approx_{F_2} C \approx_{F_2} A \cup B \approx_{F_2} B \cup C \approx_{F_2} C \cup A.$$

For example, for $X = \mathbb{S}^{n-1} = \{\vec{v} \in \mathbb{R}^n : \|\vec{v}\| = 1\}$ and $G = SO_n(\mathbb{R}) = \{\varphi \in \mathrm{Mat}(n, n, \mathbb{R}) : {}^t\varphi = \varphi^{-1}, \det\varphi = 1\}$, we have the following theorems.

Example A. (by Dekker [Dek2; W: Th.5.2], Deligne & Sullivan [DelSu], Borel [Bo])

$n \geqq 4$: even integer
$\Rightarrow$ *$^\exists F_2 \subsetneqq SO_n(\mathbb{R})$: a free subgroup such that the action $F_2 \curvearrowright \mathbb{S}^{n-1}$ is without fixed points.*

Example B. (by Świerczkowski [Ś; W: Th.2.1], Dekker [Dek2])

$n \geqq 3$: odd integer
$\Rightarrow$ *$^\exists F_2 \subsetneqq SO_n(\mathbb{R})$: a free subgroup such that the action $F_2 \curvearrowright \mathbb{S}^{n-1}$ is locally commutative.*

The rational versions for the group $SO_n(\mathbb{Q}) = SO_n(\mathbb{R}) \cap \mathrm{Mat}(n, n, \mathbb{Q})$ were conjectured by Mycielski:

Problem A.

$n \geqq 4$: even integer
$\Rightarrow$ *$^\exists F_2 \subsetneqq SO_n(\mathbb{Q})$: a free subgroup such that the action $F_2 \curvearrowright \mathbb{S}^{n-1}$ is without fixed points.*

Problem B.

$n \geqq 3$: odd integer
$\Rightarrow$ *$^\exists F_2 \subsetneqq SO_n(\mathbb{Q})$: a free subgroup such that*
the action $F_2 \curvearrowright \mathbb{S}^{n-1}$ is locally commutative and
the action $F_2 \curvearrowright \mathbb{S}^{n-1} \cap \mathbb{Q}^n = \{\vec{v} \in \mathbb{Q}^n : \|\vec{v}\| = 1\}$ is without fixed points.

Problem B was generalized by the author.

Problem B'.

$n \geqq 3$: odd integer, $q \in \mathbb{Q}$, $q \gneqq 0$
$\Rightarrow$ *$^\exists F_2 \subsetneqq SO_n(\mathbb{Q})$: a free subgroup such that*
the action $F_2 \curvearrowright \sqrt{q}\mathbb{S}^{n-1} = \{\vec{v} \in \mathbb{R}^n : \|\vec{v}\| = \sqrt{q}\}$ is locally commutative and
the action $F_2 \curvearrowright (\sqrt{q}\mathbb{S}^{n-1}) \cap \mathbb{Q}^n = \{\vec{v} \in \mathbb{Q}^n : \|\vec{v}\| = \sqrt{q}\}$ is without fixed points.

Remark. The motivation of the rational sphere version is to expect to prove the following:

- stronger results than the complete sphere version,
- the Hausdorff-Banach-Tarski paradox without the axiom of choice.

It is enough to prove them for $n = 3, 4, 5$ and 6, because Problem A for even $n + n'$ is proved by $\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta' \end{pmatrix} \rangle$ if Problem A for even n and even n' are proved by $\langle \alpha, \beta \rangle$ and $\langle \alpha', \beta' \rangle$ respectively, and Problem B' for odd $n+n'$ is proved by $\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta' \end{pmatrix} \rangle$ if Problem A for even n and Problem B' for odd n' are proved by $\langle \alpha, \beta \rangle$ and $\langle \alpha', \beta' \rangle$ respectively. We already proved them partly.

	$\sqrt{q} \in \mathbb{Q}$	$\sqrt{q} \notin \mathbb{Q}$
Problem B' for $n = 3$	shown affirmatively [Sa0]	shown affirmatively [Sa2]
Problem A for $n = 4$	shown affirmatively [Sa1]	
Problem B' for $n = 5$	not yet	shown affirmatively [Sa3]
Problem A for $n = 6$	not yet	

Theorem. [Sa0, Sa1, Sa2, Sa3] *We can prove affirmatively Problem A for $n = 4$, Problem B' for $n = 3$ and for $n = 5$, $\sqrt{q} \notin \mathbb{Q}$.*

Remark. The author believes that we can prove the remained cases, Problem A for $n = 6$ and Problem B' for $n = 5$, $\sqrt{q} \in \mathbb{Q}$.

In this conference, the author talked about [Sa3], the case of $n = 5$ and $\sqrt{q} \notin \mathbb{Q}$.

Outline of the proof.

- We can assume that $q \in \mathbb{N} \setminus \{0, 1\}$ and $\neg \exists d \in \mathbb{N} \setminus \{0, 1\}$ s.t. $d^2 \mid q$.
- We can fix a prime $\exists p$ s.t. $\left(\frac{q}{p}\right) = -1$ and $\left(\frac{-1}{p}\right) = 1$ because of Satz 147 of [H] (or [Sa2]), which implies Dirichlet's prime number theorem.
- We can fix $\exists b \in \mathbb{Z}$ s.t. $p \mid 1 + b^2$.
- Let

$$\alpha = \frac{1}{1+b^2} \begin{pmatrix} 1+b^2 & 0 & 0 & 0 & 0 \\ 0 & 1-b^2 & -2b & 0 & 0 \\ 0 & 2b & 1-b^2 & 0 & 0 \\ 0 & 0 & 0 & 1-b^2 & -2b \\ 0 & 0 & 0 & 2b & 1-b^2 \end{pmatrix} \in SO_5(\mathbb{Q}),$$

and

$$\beta = \frac{1}{1+b^2} \begin{pmatrix} 1-b^2 & -2b & 0 & 0 & 0 \\ 2b & 1-b^2 & 0 & 0 & 0 \\ 0 & 0 & 1-b^2 & -2b & 0 \\ 0 & 0 & 2b & 1-b^2 & 0 \\ 0 & 0 & 0 & 0 & 1+b^2 \end{pmatrix} \in SO_5(\mathbb{Q}).$$

Then we can prove that the group $F_2 = \langle \alpha, \beta \rangle$ satisfies required condition.

- **Lemma 0 & Corollary 1.**

$m \in \mathbb{N}$.

$$\phi = \begin{pmatrix} \phi_0^0 & \cdots & \phi_{2m}^0 \\ \vdots & \ddots & \vdots \\ \phi_0^{2m} & \cdots & \phi_{2m}^{2m} \end{pmatrix} \in SO_{2m+1}(\mathbb{R}),$$

$\vec{\mathrm{ax}}(\phi) \neq \vec{0}$

$\Rightarrow$ $\{\vec{v} \in \mathbb{R}^{2m+1} : \phi(\vec{v}) = \vec{v}\} = \{a \cdot \vec{\mathrm{ax}}(\phi) : a \in \mathbb{R}\}$, *where*

$$\vec{\mathrm{ax}}(\phi) \underset{\mathrm{def}}{=} \frac{1}{2^m m!} \begin{pmatrix} \vdots \\ \vdots \\ \displaystyle\sum_{\mathrm{s} \in \mathrm{S}_{2m}} \mathrm{sgn}\,\mathrm{s} \prod_{r=0}^{m-1} \left(\phi_{(i+1+\mathrm{s}(2r)) \bmod (2m+1)}^{(i+1+\mathrm{s}(2r+1)) \bmod (2m+1)} - \phi_{(i+1+\mathrm{s}(2r+1)) \bmod (2m+1)}^{(i+1+\mathrm{s}(2r)) \bmod (2m+1)}\right) \\ \vdots \\ \vdots \end{pmatrix} \begin{matrix} (0) \\ \vdots \\ (i) \\ \vdots \\ (2m) \end{matrix}$$

and $\mathfrak{S}_{2m} = \{\mathfrak{s} : \{0,1,\ldots,2m-1\} \to \{0,1,\ldots,2m-1\},\ \text{bijection}\}$, *for example,*

$$\phi \in SO_5(\mathbb{R}) \Rightarrow \vec{\mathrm{ax}}(\phi) \underset{\text{def}}{=} \begin{pmatrix} (\phi_1^2-\phi_2^1)(\phi_3^4-\phi_4^3)-(\phi_1^3-\phi_3^1)(\phi_2^4-\phi_4^2)+(\phi_1^4-\phi_4^1)(\phi_2^3-\phi_3^2) \\ (\phi_2^3-\phi_3^2)(\phi_4^0-\phi_0^4)-(\phi_2^4-\phi_4^2)(\phi_3^0-\phi_0^3)+(\phi_2^0-\phi_0^2)(\phi_3^4-\phi_4^3) \\ (\phi_3^4-\phi_4^3)(\phi_0^1-\phi_1^0)-(\phi_3^0-\phi_0^3)(\phi_4^1-\phi_1^4)+(\phi_3^1-\phi_1^3)(\phi_4^0-\phi_0^4) \\ (\phi_4^0-\phi_0^4)(\phi_1^2-\phi_2^1)-(\phi_4^1-\phi_1^4)(\phi_0^2-\phi_2^0)+(\phi_4^2-\phi_2^4)(\phi_0^1-\phi_1^0) \\ (\phi_0^1-\phi_1^0)(\phi_2^3-\phi_3^2)-(\phi_0^2-\phi_2^0)(\phi_1^3-\phi_3^1)+(\phi_0^3-\phi_3^0)(\phi_1^2-\phi_2^1) \end{pmatrix}.$$

- **Lemmas 1 & 2.**
  $^\forall w \in F_2 \setminus \{\mathrm{id}\},\ ^\exists M \in \mathbb{N}\setminus\{0\},\ ^\exists P, Q, R, S \in \mathbb{Z}$: *such that*
  $w = \alpha^{\varepsilon'} \cdots \alpha^{\varepsilon} \Rightarrow$

$PS - QR \equiv 4^{M-1} \pmod p$,

$$(1+b^2)^{\sharp w} w \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & P & -\varepsilon Pb & R & -\varepsilon Rb \\ 0 & \varepsilon' Pb & -\varepsilon'\varepsilon Pb^2 & \varepsilon' Rb & -\varepsilon'\varepsilon Rb^2 \\ 0 & Q & -\varepsilon Qb & S & -\varepsilon Sb \\ 0 & \varepsilon' Qb & -\varepsilon'\varepsilon Qb^2 & \varepsilon' Sb & -\varepsilon'\varepsilon Sb^2 \end{pmatrix},\ \text{so } (1+b^2)^{2\cdot\sharp w}\vec{\mathrm{ax}}(w) \equiv -4^M \begin{pmatrix} (1+\varepsilon'\varepsilon)/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$w = \alpha^{\varepsilon'} \cdots \beta^{\delta} \Rightarrow$

$PS - QR \equiv -4^{M} \pmod p$,

$$(1+b^2)^{\sharp w} w \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ P & -\delta Pb & R & -\delta Rb & 0 \\ \varepsilon' Pb & -\varepsilon'\delta Pb^2 & \varepsilon' Rb & -\varepsilon'\delta Rb^2 & 0 \\ Q & -\delta Qb & S & -\delta Sb & 0 \\ \varepsilon' Qb & -\varepsilon'\delta Qb^2 & \varepsilon' Sb & -\varepsilon'\delta Sb^2 & 0 \end{pmatrix},\ \text{so } (1+b^2)^{2\cdot\sharp w}\vec{\mathrm{ax}}(w) \equiv -4^M \begin{pmatrix} 1 \\ -\delta b \\ \varepsilon'\delta \\ -\varepsilon' b \\ 1 \end{pmatrix},$$

$w = \beta^{\delta'} \cdots \alpha^{\varepsilon} \Rightarrow$

$PS - QR \equiv -4^{M} \pmod p$,

$$(1+b^2)^{\sharp w} w \equiv \begin{pmatrix} 0 & P & -\varepsilon Pb & R & -\varepsilon Rb \\ 0 & \delta' Pb & -\delta'\varepsilon Pb^2 & \delta' Rb & -\delta'\varepsilon Rb^2 \\ 0 & Q & -\varepsilon Qb & S & -\varepsilon Sb \\ 0 & \delta' Qb & -\delta'\varepsilon Qb^2 & \delta' Sb & -\delta'\varepsilon Sb^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ \text{so } (1+b^2)^{2\cdot\sharp w}\vec{\mathrm{ax}}(w) \equiv -4^M \begin{pmatrix} 1 \\ \delta' b \\ \delta'\varepsilon \\ \varepsilon b \\ 1 \end{pmatrix}.$$

$w = \beta^{\delta'} \cdots \beta^{\delta} \Rightarrow$

$PS - QR \equiv 4^{M-1} \pmod p$,

$$(1+b^2)^{\sharp w} w \equiv \begin{pmatrix} P & -\delta Pb & R & -\delta Rb & 0 \\ \delta' Pb & -\delta'\delta Pb^2 & \delta' Rb & -\delta'\delta Rb^2 & 0 \\ Q & -\delta Qb & S & -\delta Sb & 0 \\ \delta' Qb & -\delta'\delta Qb^2 & \delta' Sb & -\delta'\delta Sb^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ \text{so } (1+b^2)^{2\cdot\sharp w}\vec{\mathrm{ax}}(w) \equiv -4^M \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (1+\delta'\delta)/2 \end{pmatrix}.$$

where $(z_j^i) \equiv (z'^i_j) \underset{\text{def}}{\Leftrightarrow} {}^\forall i, {}^\forall j,\ z_j^i \equiv z'^i_j \pmod p$ *and* $(z_i) \equiv (z'_i) \underset{\text{def}}{\Leftrightarrow} {}^\forall i,\ z_i \equiv z'_i \pmod p$.

- **Corollary 2.** From Corollary 1 and Lemma 2, $F_2 = \langle \alpha, \beta \rangle$ is a free group and $\dim\{\vec{v} \in \mathbb{R}^5 : w(\vec{v}) = \vec{v}\} = 1$ for $w \in F_2 \setminus \{\mathrm{id}\}$.
- Proof of "the action $F_2 \curvearrowright (\sqrt{q}\mathbb{S}^4) \cap \mathbb{Q}^5$ is without fixed points".
  It is enough to show that $^\forall w \in F_2 \setminus \{\mathrm{id}\}$,

(the first letter of $w)^{-1} \neq$ (the last letter of $w) \Rightarrow \neg{}^\exists \vec{v} \in (\sqrt{q}\mathbb{S}^4) \cap \mathbb{Q}^5$ s.t. $w(\vec{v}) = \vec{v}$.
(w is said to be cyclically reduced)

because $\vec{\mathrm{ax}}(\lambda\varpi\lambda^{-1}) = \lambda(\vec{\mathrm{ax}}(\varpi))$. For cyclically reduced w,

$$\|\vec{\mathrm{ax}}(w)\| / \sqrt{q} \notin \mathbb{Q}$$

from $q \cdot (1+b^2)^{4 \cdot \sharp w} \|\vec{\mathrm{ax}}(w)\|^2 \equiv q \cdot 16^M \pmod{p}$ by Lemma 2. So the intersection points of the axis of w and the complete sphere $\sqrt{q}\mathbb{S}^4$.

$$\pm\sqrt{q}\frac{\vec{\mathrm{ax}}(w)}{\|\vec{\mathrm{ax}}(w)\|}$$

are not included in $\mathbb{Q}^5$. □

- Let

$$\begin{aligned} w \sim w' &\underset{\text{def}}{\Leftrightarrow} {}^{\exists}\vec{v} \in \sqrt{q}\mathbb{S}^4 : \text{ s.t. } w(\vec{v}) = \vec{v} = w'(\vec{v}), \\ w \simeq w' &\underset{\text{def}}{\Leftrightarrow} ww' = w'w. \end{aligned}$$

Then $\sim$ and $\simeq$ are equivalence relations on $F_2 \setminus \{\mathrm{id}\}$ which satisfy

$$\left.\begin{aligned} w^k \sim w'^l \Leftrightarrow w \sim w' \Leftrightarrow \varpi w \varpi^{-1} \sim \varpi w' \varpi^{-1} \\ w^k \simeq w'^l \Leftrightarrow w \simeq w' \Leftrightarrow \varpi w \varpi^{-1} \simeq \varpi w' \varpi^{-1} \end{aligned}\right. \quad \text{for } {}^{\forall}k, {}^{\forall}l \in \mathbb{Z} \setminus \{0\},$$

$$\left.\begin{aligned} w \sim w'w \Leftrightarrow w \sim w' \Leftrightarrow w \sim ww' \\ w \simeq w'w \Leftrightarrow w \simeq w' \Leftrightarrow w \simeq ww' \end{aligned}\right. \quad \text{if } w^{-1} \neq w'.$$

- **Lemma 3.**

w, $w' \in F_2 \setminus \{\mathrm{id}\}$ *of distinct types of the following six kind,*

$$\begin{matrix} \alpha \cdots \alpha, & \alpha^{-1} \cdots \beta^{-1}, & \alpha^{-1} \cdots \beta, \\ \beta \cdots \beta, & \alpha \cdots \beta^{-1}, & \alpha \cdots \beta, \end{matrix}$$

$\Rightarrow$ $w \not\sim w'$.

Proof. Obvious from Lemma 2. □

- **Lemma 4.**

w, $w' \in F_2 \setminus \{\mathrm{id}\}$ *of same type of the above kind*

$\Rightarrow$ $w \sim w'$.

Proof. We denote $w \subseteq w' \underset{\text{def}}{\Leftrightarrow} {}^{\exists}\tilde{w}$ s.t. $w\tilde{w} = w'$, without cancellation.
Let κ and λ be of $\{\alpha^{-1}, \beta^{-1}, \alpha, \beta\}$ such that $w = \kappa \cdots \lambda$ and $w' = \kappa \cdots \lambda$. Then $\kappa^{-1} \neq \lambda$.
If $w = \underbrace{\kappa \cdots \sigma}_{\varpi}\underbrace{\tau \cdots \lambda}_{\hat{w}}$ and $w' = \underbrace{\kappa \cdots \sigma}_{\varpi}\underbrace{\tau' \cdots \lambda}_{\hat{w}'}$ $(\tau \neq \tau')$ then $\varpi^{-1} w \varpi = \underbrace{\tau \cdots \lambda}_{\hat{w}}\underbrace{\kappa \cdots \sigma}_{\varpi} \not\sim \underbrace{\tau' \cdots \lambda}_{\hat{w}'}\underbrace{\kappa \cdots \sigma}_{\varpi} = \varpi^{-1} w' \varpi$. a contradiction. So $w \subseteq w'$ or $w \supseteq w'$. We can assume $w \subseteq w'$.
If $w \neq w'$ then $w\tilde{w} = \underbrace{\kappa \cdots \lambda}_{w}\underbrace{\kappa' \cdots \lambda}_{\tilde{w}} = w'$ without cancellation (so $\kappa'^{-1} \neq \lambda$). So $\kappa \cdots \lambda = w \sim \tilde{w} = \kappa' \cdots \lambda$. By Lemma 3, $\tilde{w} = \kappa \cdots \lambda$. It reduces the proof for w and $\tilde{w}$.
Hence, by induction, we can assume $w = w'$. $w \simeq w'$ is obvious. □

- **Lemma 5.**

$w = \alpha^{\varepsilon} \cdots \beta^{\delta}$, *either* $w' = \alpha^{\varepsilon} \cdots \alpha^{-\varepsilon}$ *or* $w' = \beta^{-\delta} \cdots \beta^{\delta}$

$\Rightarrow$ $w \not\sim w'$.

Proof. For $w' = \alpha^\varepsilon \cdots \alpha^{-\varepsilon}$,
if $w \subseteq w'$,

$$w' = \underbrace{\alpha^\varepsilon \cdots \beta^\delta}_{w} \alpha^\varepsilon \cdots \alpha^{-\varepsilon} \Rightarrow \text{it reduces the proof for } w \text{ and } w^{-1}w' = \alpha^\varepsilon \cdots \alpha^{-\varepsilon},$$

$$w' = \underbrace{\alpha^\varepsilon \cdots \beta^\delta}_{w} \alpha^{-\varepsilon} \cdots \alpha^{-\varepsilon} \Rightarrow w = \alpha^\varepsilon \cdots \beta^\delta \not\sim \alpha^\varepsilon \cdots \alpha^\varepsilon = (w^{-1}w')^{-1}. \qquad \text{so } w \not\sim w'.$$

$$w' = \underbrace{\alpha^\varepsilon \cdots \beta^\delta}_{w} \beta^\delta \cdots \alpha^{-\varepsilon} \Rightarrow w = \alpha^\varepsilon \cdots \beta^\delta \not\sim \alpha^\varepsilon \cdots \beta^{-\delta} = (w^{-1}w')^{-1}. \qquad \text{so } w \not\sim w'.$$

If $w \supseteq w'$ (so neither $w \supseteq w'^{-1}$ nor $w \subseteq w'^{-1}$),

$$\alpha^\varepsilon \cdots \beta^\delta = w \not\simeq w'w = \alpha^\varepsilon \cdots \underbrace{\cdots \alpha^{-\varepsilon}\alpha^\varepsilon \cdots}_{\text{cancellation}} \cdots \beta^\delta. \qquad \text{By Lemma 4, } w \not\sim w'w.$$

If neither $w \supseteq w'$ nor $w \subseteq w'$,

$$\alpha^\varepsilon \cdots \beta^\delta = w \not\simeq w'^{-1}w = \alpha^\varepsilon \cdots \underbrace{\cdots \alpha^{-\varepsilon}\alpha^\varepsilon \cdots}_{\text{cancellation}} \cdots \beta^\delta. \qquad \text{By Lemma 4, } w \not\sim w'^{-1}w.$$

For $w' = \beta^{-\delta} \cdots \beta^\delta$, similar. □

- Proof of "the action $F_2 \curvearrowright \sqrt{q}\mathbb{S}^4$ is locally commutative", that is, "$w \sim w' \Rightarrow w \simeq w'$".
It is enough to show it for $w = \alpha$, $w = \beta$ and $w = \alpha^\varepsilon \cdots \beta^\delta$. Let $w' = \lambda' \cdots \lambda$. Then, for $w = \alpha$,

	$\lambda = \alpha^{-1}$	$\lambda = \beta^{-1}$	$\lambda = \alpha$	$\lambda = \beta$
$\lambda' = \alpha^{-1}$	$w \sim w'^{-1} \Rightarrow w \simeq w'^{-1}$ (4)	$w \not\sim w'$ (3)	$w \not\sim w'w^{-k}$ (3)	$w \not\sim w'$ (3)
$\lambda' = \beta^{-1}$	$w \not\sim w'^{-1}$ (3)	$w \not\sim w'^{-1}$ (3)	$w \not\sim w'^{-1}$ (3)	$w \not\sim ww'$ (3)
$\lambda' = \alpha$	$w \not\sim w'w^k$ (3)	$w \not\sim w'$ (3)	$w \sim w' \Rightarrow w \simeq w'$ (4)	$w \not\sim w'$ (3)
$\lambda' = \beta$	$w \not\sim w'^{-1}$ (3)	$w \not\sim ww'$ (3)	$w \not\sim w'^{-1}$ (3)	$w \not\sim w'$ (3)

where $w' = \alpha^{-1} \cdots \beta^{\pm 1}\alpha^k$ for $w = \alpha^{-1} \cdots \alpha$, $w' = \alpha \cdots \beta^{\pm 1}\alpha^{-k}$ for $w = \alpha \cdots \alpha^{-1}$. For $w = \beta$, similar. For $w = \alpha^\varepsilon \cdots \beta^\delta$.

	$\lambda = \alpha^{-\varepsilon}$	$\lambda = \beta^{-\delta}$	$\lambda = \alpha^\varepsilon$	$\lambda = \beta^\delta$
$\lambda' = \alpha^{-\varepsilon}$	$w \not\sim w'^{-1}$ (3)	$w \not\sim w'$ (3)	$w \not\sim ww'$ (3)	$w \not\sim w'$ (3)
$\lambda' = \beta^{-\delta}$	$w \sim w'^{-1} \Rightarrow w \simeq w'^{-1}$ (4)	$w \not\sim w'^{-1}$ (3)	$w \not\sim w'^{-1}$ (3)	$w \not\sim w'$ (5)
$\lambda' = \alpha^\varepsilon$	$w \not\sim w'$ (5)	$w \not\sim w'$ (3)	$w \not\sim w'$ (3)	$w \sim w' \Rightarrow w \simeq w'$ (4)
$\lambda' = \beta^\delta$	$w \not\sim w'^{-1}$ (3)	$w \not\sim ww'$ (3)	$w \not\sim w'^{-1}$ (3)	$w \not\sim w'$ (3)

where (z) means that the proof requires Lemma z. □

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